

Strong submultiplicativity of the Poincaré metric

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To David Minda on the occasion of his retirement

Abstract

We give a direct proof of an important result of Solynin which says that the Poincaré metric is a strongly submultiplicative domain function. This result is then used to define a new capacity for compact subsets of the complex plane \mathbb{C} , which might be called Poincaré capacity. If the compact set $K \subseteq \mathbb{C}$ is connected, then the Poincaré capacity of K is the same as the logarithmic capacity of K . In this special case, the submultiplicativity is well-known and can be stated as an inequality for the normalized conformal map onto the complement of K . Using the connection between Poincaré metrics and universal covering maps this inequality is extended to the much wider class of universal covering maps.

1 Introduction

Let Ω be an open subset of the Riemann sphere $\hat{\mathbb{C}}$ and suppose that Ω is *hyperbolic* i.e., $\hat{\mathbb{C}} \setminus \Omega$ contains at least three pairwise different points. Then Ω carries a unique complete conformal Riemannian metric $\lambda_\Omega(z) |dz|$ with Gaussian curvature -1 , the so-called hyperbolic metric or Poincaré metric of Ω . That $\lambda_\Omega(z) |dz|$ has constant curvature -1 is equivalent to the fact that in local coordinates the hyperbolic density $\lambda_\Omega(z)$ satisfies the nonlinear elliptic PDE

$$\Delta \log \lambda_\Omega(z) = \lambda_\Omega(z)^2. \quad (1.1)$$

Only in very rare cases it is possible to give an explicit formula for the hyperbolic metric (see [1]). It is therefore of interest to give good qualitative estimates for the Poincaré metric, see e.g. [4, 11, 26] and the more recent references [3, 5, 7, 14, 23, 24].

The first aim of this paper is to give a full and direct proof of the following beautiful sharp inequality due to Solynin [21, 22] which relates the Poincaré metrics of two hyperbolic *domains* with the Poincaré metric of their union and their intersection.

Theorem 1.1

Let Ω_1 and Ω_2 be domains in $\hat{\mathbb{C}}$ such that $\Omega_1 \cap \Omega_2 \neq \emptyset$. Suppose that $\Omega_1 \cup \Omega_2$ is hyperbolic. Then

$$\frac{\lambda_{\Omega_1}(z) \cdot \lambda_{\Omega_2}(z)}{\lambda_{\Omega_1 \cup \Omega_2}(z) \cdot \lambda_{\Omega_1 \cap \Omega_2}(z)} \geq 1 \quad \text{for all } z \in \Omega_1 \cap \Omega_2. \quad (1.2)$$

If equality holds for one point $z \in \Omega_1 \cap \Omega_2$, then $\Omega_1 \subseteq \Omega_2$ or $\Omega_2 \subseteq \Omega_1$. In this case, equality holds for all points in $\Omega_1 \cap \Omega_2$.

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Remark 1.2

It might be worth mentioning that in general it makes no sense to speak of the value of the density of a conformal metric at a specific point on the Riemann sphere, but it makes sense to speak of the value of the *quotient* of two such metrics, see [15, p. 57], so the left hand side of inequality (1.2) as the product of two such quotients is indeed meaningful.

Remark 1.3

As noted above, Theorem 1.1 is due to Solynin [21, 22] who even obtained an extended form of inequality (1.2) for finitely many domains instead of just two domains, see [21, Theorem 2]. In fact, Solynin obtained the estimate (1.2) as a corollary to a much more general comparison result for solutions for a certain class of nonlinear elliptic PDEs. The proof we give below is somewhat similar, but more direct and the details are different as we only use the classical maximum principle for subharmonic functions. In order to be able to make use of the classical maximum principle we first establish in Lemma 2.7 a preliminary *nonsharp* form of the inequality (1.2) which can be proved analogously to the classical Ahlfors lemma [2]. What makes the proof of Lemma 2.7 work is the fact that it is very useful to multiply conformal metrics. This is just one of the many beautiful insights of David Minda about conformal metrics which we have tried to learn from him. In order to treat the cases of equality in (1.2) we proceed along the lines of David's paper [16] which is concerned with the case of equality in Ahlfors' lemma.

As pointed out by Solynin [21, 22], Theorem 1.1 can be used to define for (almost) all compact subsets K of a hyperbolic domain $\Omega \subseteq \hat{\mathbb{C}}$ a “capacity” in terms of the Poincaré metrics $\lambda_{\Omega \setminus K}(z) |dz|$ and $\lambda_{\Omega}(z) |dz|$ which depends on the “ambient” domain Ω , see Remark 1.5 below for details. In the following, we propose a different definition which assigns to each compact subset of the complex plane \mathbb{C} a capacity. We slightly abuse notation and denote by $\lambda_{\hat{\mathbb{C}}}(z) |dz|$ the *spherical* metric on the Riemann sphere $\hat{\mathbb{C}}$, that is, the unique conformal metric on $\hat{\mathbb{C}}$ with constant curvature $+1$.

Definition 1.4

Let K be a compact subset of \mathbb{C} . If K contains at least three pairwise different points, then we set

$$\text{pcap}(K) := \frac{\lambda_{\hat{\mathbb{C}} \setminus K}(\infty)}{\lambda_{\hat{\mathbb{C}}}(\infty)}.$$

If K contains at most two different points, we set $\text{pcap}(K) := 0$. We call the number $\text{pcap}(K)$ the *Poincaré capacity* of the compact set K .

Note that the definition of $\text{pcap}(K)$ is meaningful since the quotient $\lambda_{\hat{\mathbb{C}} \setminus K} / \lambda_{\hat{\mathbb{C}}}$ has a well-defined value at the interior point ∞ of $\hat{\mathbb{C}} \setminus K$ for every compact set $K \subseteq \mathbb{C}$ with at least three pairwise different points.

Remark 1.5

As noted above, Solynin's approach [21, 22] of relating the Poincaré metric of a domain $\Omega \subseteq \hat{\mathbb{C}}$ with a “capacity” of the complement of Ω is different from Definition 1.4: Solynin fixes a hyperbolic domain $\Omega \subseteq \hat{\mathbb{C}}$ and a point $z_0 \in \Omega$, and considers (the logarithm of)

$$C_{\Omega, z_0}(K) := \frac{\lambda_{\Omega \setminus K}(z_0)}{\lambda_{\Omega}(z_0)}$$

for compact subsets K of $\Omega \setminus \{z_0\}$. Note that $C_{\Omega, z_0}(K)$ depends not only on K , but also on the “ambient” domain Ω and the point z_0 whereas $\text{pcap}(K)$ only depends on K . One of the main advantages of the definition of pcap is the fact that it relates directly to universal covering maps in the same way as logarithmic capacity relates to conformal maps (see Remark 1.7 below).

Theorem 1.1 immediately implies the following result.

Corollary 1.6

Let K_1 and K_2 be compact subsets of \mathbb{C} . Then

$$\text{pcap}(K_1 \cup K_2) \cdot \text{pcap}(K_1 \cap K_2) \leq \text{pcap}(K_1) \cdot \text{pcap}(K_2).$$

In particular, we have the *strong subadditivity property*

$$\log \text{pcap}(K_1 \cup K_2) \leq \log \text{pcap}(K_1) + \log \text{pcap}(K_2) - \log \text{pcap}(K_1 \cap K_2)$$

in the sense of Choquet's general theory of capacities [6] provided that we interpret this inequality with care in the case when $K_1 \cap K_2$ contains at most two distinct points. Therefore, the inequality of Theorem 1.1, which in local coordinates takes the form

$$\lambda_{\Omega_1}(z) \cdot \lambda_{\Omega_2}(z) \geq \lambda_{\Omega_1 \cup \Omega_2}(z) \cdot \lambda_{\Omega_1 \cap \Omega_2}(z),$$

can be viewed as a *strong submultiplicativity property* of the Poincaré metric. We note that the analogous result for the *logarithmic capacity* $\text{cap}(K)$ of a compact set K (see [19, Chapter 5]), i.e.,

$$\log \text{cap}(K_1 \cup K_2) \leq \log \text{cap}(K_1) + \log \text{cap}(K_2) - \log \text{cap}(K_1 \cap K_2) \quad (1.3)$$

is a well-known inequality in potential theory (see e.g. [20] for a proof).

Remark 1.7 (Poincaré capacity by way of universal covering maps)

It is instructive to point out an alternate way of defining the Poincaré capacity based on universal covering maps. Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disk, let $K \subseteq \mathbb{C}$ be a compact set with at least three pairwise different points, and denote by π_K the universal covering map from $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ onto $\hat{\mathbb{C}} \setminus K$ normalized in such a way that $\pi_K(\infty) = \infty$ and

$$\pi'_K(\infty) := \lim_{z \rightarrow \infty} \frac{\pi_K(z)}{z} > 0.$$

Then π_K has an expansion at ∞ of the form

$$\pi_K(z) = \pi'_K(\infty)z + \sum_{k=0}^{\infty} a_k z^{-k}, \quad z \in \mathbb{C} \setminus \overline{\mathbb{D}}. \quad (1.4)$$

Recall that $\lambda_{\hat{\mathbb{C}} \setminus K}(z) |dz|$ and $\lambda_{\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}}(z) |dz|$ are related via π_K as follows

$$\lambda_{\hat{\mathbb{C}} \setminus K}(\pi_K(z)) |\pi'_K(z)| = \lambda_{\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}}(z), \quad z \in \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}.$$

Since in local coordinates

$$\lambda_{\hat{\mathbb{C}}}(z) = \frac{2}{1 + |z|^2} \quad \text{and} \quad \lambda_{\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}}(z) = \frac{2}{|z|^2 - 1},$$

we therefore get that

$$\begin{aligned} \text{pcap}(K) &= \frac{\lambda_{\hat{\mathbb{C}} \setminus K}(\infty)}{\lambda_{\hat{\mathbb{C}}}(\infty)} = \lim_{z \rightarrow \infty} \frac{\lambda_{\hat{\mathbb{C}} \setminus K}(\pi_K(z)) |\pi'_K(z)|}{\lambda_{\hat{\mathbb{C}}}(\pi_K(z)) |\pi'_K(z)|} = \lim_{z \rightarrow \infty} \frac{\lambda_{\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}}(z)}{\lambda_{\hat{\mathbb{C}}}(\pi_K(z)) |\pi'_K(z)|} \\ &= \lim_{z \rightarrow \infty} \frac{1}{|z|^2 - 1} \cdot \frac{1 + |\pi_K(z)|^2}{|\pi'_K(z)|} \stackrel{(1.4)}{=} \pi'_K(\infty). \end{aligned}$$

In view of this remark, Theorem 1.1 can be rephrased in the following way.

Corollary 1.8

Let K_1 and K_2 be compact subsets of \mathbb{C} such that $K_1 \cap K_2$ contains at least three pairwise different points. Then

$$\log \pi'_{K_1 \cup K_2}(\infty) \leq \log \pi'_{K_1}(\infty) + \log \pi'_{K_2}(\infty) - \log \pi'_{K_1 \cap K_2}(\infty). \quad (1.5)$$

In the special case when K is connected, its complement $\hat{\mathbb{C}} \setminus K$ is a simply connected domain. Hence the universal covering map π_K is a conformal map from $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ onto $\hat{\mathbb{C}} \setminus K$. In this case the logarithmic capacity $\text{cap}(K)$ can be computed as $\text{cap}(K) = \pi'_K(\infty)$, see e.g. [18, Corollary 9.9]. Combined with Remark 1.7 this leads to the following result.

Corollary 1.9 (Poincaré capacity vs. logarithmic capacity)

Let K be a compact and connected subset of \mathbb{C} . Then $\text{pcap}(K) = \text{cap}(K)$.

In particular, if K_1 , K_2 and $K_1 \cap K_2$ are connected compact sets in \mathbb{C} , then π_{K_1} , π_{K_2} , $\pi_{K_1 \cap K_2}$ and $\pi_{K_1 \cup K_2}$ are all conformal maps. In this case, (1.5) is a well-known inequality for conformal maps, which follows for instance immediately from (1.3). Therefore, Corollary 1.8 shows that the same inequality is in fact true for the much wider class of universal covering maps. Finally, we note that if $\hat{\mathbb{C}} \setminus K$ is not simply connected, then in general $\text{pcap}(K) \neq \text{cap}(K)$. For instance, take a finite set K with at least three pairwise different points. Then $\text{cap}(K) = 0$, but $\text{pcap}(K) > 0$.

2 Proof of Theorem 1.1

We will prove a slight extension of Theorem 1.1 by allowing open sets instead of domains. It is clearly sufficient to restrict consideration to subsets of \mathbb{C} . For this purpose, we denote for an open set $U \subseteq \mathbb{C}$ and a point $z \in U$ by $U(z)$ the connected component (i.e., the maximal open connected subset) of U which contains the point z .

Theorem 2.1

Let Ω_1 and Ω_2 be open sets in \mathbb{C} such that $\Omega_1 \cap \Omega_2 \neq \emptyset$ and $\Omega_1 \cup \Omega_2$ is hyperbolic. Then

$$\lambda_{\Omega_1}(z) \cdot \lambda_{\Omega_2}(z) \geq \lambda_{\Omega_1 \cup \Omega_2}(z) \cdot \lambda_{\Omega_1 \cap \Omega_2}(z) \quad \text{for all } z \in \Omega_1 \cap \Omega_2. \quad (2.1)$$

If equality holds for one point $z \in \Omega_1 \cap \Omega_2$, then $\Omega_1(z) \subseteq \Omega_2(z)$ or $\Omega_2(z) \subseteq \Omega_1(z)$. In this case, equality holds for all points in $\Omega_1(z) \cap \Omega_2(z)$.

The proof of Theorem 2.1 will occupy the rest of this paper. We first recall the well-known *monotonicity property of the hyperbolic metric*.

Lemma 2.2

Let $\Omega \subseteq \mathbb{C}$ be a hyperbolic open set and let U be an open subset of Ω . Then

$$\lambda_{\Omega}(z) \leq \lambda_U(z) \quad \text{for every } z \in U. \quad (2.2)$$

In particular,

$$\lim_{z \rightarrow \xi} \lambda_{\Omega}(z) = +\infty \quad \text{for every } \xi \in \partial\Omega. \quad (2.3)$$

Proof. The estimate (2.2) follows directly from the definition of the hyperbolic metric as the *maximal* conformal metric with curvature -1 . In order to prove (2.3) let ξ, η be two different boundary points of Ω , so $\Omega \subseteq \mathbb{C} \setminus \{\xi, \eta\}$. Then

$$\lim_{z \rightarrow \xi} \lambda_{\mathbb{C} \setminus \{\xi, \eta\}}(z) = +\infty,$$

see e.g. [11, formula (4.1)]. Since $\lambda_{\Omega}(z) \geq \lambda_{\mathbb{C} \setminus \{\xi, \eta\}}(z)$ by (2.2), we deduce $\lambda_{\Omega}(z) \rightarrow +\infty$ as $z \rightarrow \xi$. \square

In order to prove Theorem 2.1, we need more precise information about the boundary behaviour of λ_{Ω} than provided by Lemma 2.2. At least for smooth open sets such information is available with the help of a boundary version of Ahlfors' lemma [2] proved in [13].

We first make precise what we mean by “smooth” open sets. We call a Jordan domain G (i.e., a domain bounded by a Jordan curve in \mathbb{C}) smooth, if there is a conformal map ϕ from \mathbb{D} onto G such that $|\phi'|$ extends continuously to $\partial\mathbb{D}$ with $|\phi'| \neq 0$ on $\partial\mathbb{D}$. By Carathéodory's extension theorem, this conformal map ϕ extends to a homeomorphism of the closures $\overline{\mathbb{D}}$ and \overline{G} .

Definition 2.3

Let $\Omega \subseteq \mathbb{C}$ be an open set. A subset Γ of the boundary of Ω is called smooth if for every point $\xi \in \Gamma$ there exists a smooth Jordan domain $G \subseteq \Omega$ and an open neighborhood $V \subseteq \mathbb{C}$ of ξ such that $\xi \in \partial G \cap V \subseteq \Gamma$. We call the open set Ω smooth, if $\partial\Omega$ is smooth.

Note that if an open set $\Omega \subseteq \mathbb{C}$ is bounded by finitely many analytic Jordan curves, then Ω is smooth.

Lemma 2.4

Let $\Omega \subseteq \mathbb{C}$ be a hyperbolic open set and let U be an open subset of Ω . Suppose that $\Gamma \subseteq \partial U \cap \partial\Omega$ is a smooth subset of the boundary of U as well as of the boundary of Ω . Then

$$\lim_{z \rightarrow \xi} \frac{\lambda_\Omega(z)}{\lambda_U(z)} = 1 \quad \text{for every } \xi \in \Gamma.$$

Proof. In view of Lemma 2.2 we have $\lambda_\Omega(z) \leq \lambda_U(z)$ for all $z \in U$, so we only need to prove

$$\liminf_{z \rightarrow \xi} \frac{\lambda_\Omega(z)}{\lambda_U(z)} \geq 1 \quad \text{for every } \xi \in \Gamma. \quad (2.4)$$

Since $\lim_{z \rightarrow \xi} \lambda_\Omega(z) = +\infty$ for every $\xi \in \Gamma \subseteq \partial\Omega$ by Lemma 2.2 and since both metrics $\lambda_\Omega(z)|dz|$ and $\lambda_U(z)|dz|$ have constant curvature -1 , we can apply the boundary Ahlfors lemma (Theorem 5.1 in [13]), and this gives (2.4). \square

It is always possible to exhaust a given hyperbolic set by smooth hyperbolic sets. This is a consequence of the following well-known result, see e.g. [25, p. 91].

Lemma 2.5

Let Ω_1 and Ω_2 be open sets in \mathbb{C} such that $\Omega_1 \cap \Omega_2 \neq \emptyset$ and $\Omega_1 \cup \Omega_2$ is hyperbolic. Then for each $n = 1, 2, \dots$ there exist smooth open subsets $\Omega_{1,n}$ of Ω_1 and $\Omega_{2,n}$ of Ω_2 such that

(a) $\Omega_{1,n}$ is compactly contained in $\Omega_{1,n+1}$ and $\Omega_{2,n}$ is compactly contained in $\Omega_{2,n+1}$ for each $n = 1, 2, \dots$;

(b) $\bigcup_{n=1}^{\infty} \Omega_{1,n} = \Omega_1$ and $\bigcup_{n=1}^{\infty} \Omega_{2,n} = \Omega_2$; and

(c) $\Omega_{1,n} \cap \Omega_{2,n} \neq \emptyset$ for every $n = 1, 2, \dots$

It is even possible to choose the open sets $\Omega_{1,n}$ and $\Omega_{2,n}$ in such a way that they are bounded by finitely many analytic Jordan curves.

Lemma 2.6

Let $\Omega \subseteq \mathbb{C}$ be a hyperbolic open set and for each $n = 1, 2, \dots$ let $\Omega_n \neq \emptyset$ be an open subset of Ω such that $\Omega_n \subseteq \Omega_{n+1}$ for $n = 1, 2, \dots$ and $\bigcup_{n=1}^{\infty} \Omega_n = \Omega$. Then for each $z \in \Omega$, we have

$$\lim_{n \rightarrow \infty} \lambda_{\Omega_n}(z) = \lambda_{\Omega}(z).$$

Proof. The monotonicity property of the hyperbolic metric shows that $\lambda_{\Omega_n}(z) \geq \lambda_{\Omega_{n+1}}(z) \geq \lambda_{\Omega}(z)$ for all $z \in \Omega$ and all (but finitely many) positive integers n . Hence the limit

$$\lambda(z) := \lim_{n \rightarrow \infty} \lambda_n(z)$$

exists for every $z \in \Omega$ and $\lambda(z) \geq \lambda_{\Omega}(z)$ for any $z \in \Omega$. By a result of Heins [9, Lemma 11.1], $\lambda(z) |dz|$ is a conformal metric on Ω with constant curvature -1 . Since $\lambda_{\Omega}(z) |dz|$ is the maximal metric with these properties, we deduce $\lambda(z) |dz| = \lambda_{\Omega}(z) |dz|$. \square

Lemma 2.6 can also be deduced from the results of [10]. We next prove Theorem 2.1 in a weak form.

Lemma 2.7

Let Ω_1 and Ω_2 be open sets in \mathbb{C} such that $\Omega_1 \cap \Omega_2 \neq \emptyset$ and $\Omega_1 \cup \Omega_2$ is hyperbolic. Then

$$\lambda_{\Omega_1}(z) \cdot \lambda_{\Omega_2}(z) \geq \frac{1}{\sqrt{2}} \cdot \lambda_{\Omega_1 \cup \Omega_2}(z) \cdot \lambda_{\Omega_1 \cap \Omega_2}(z) \quad \text{for all } z \in \Omega_1 \cap \Omega_2. \quad (2.5)$$

Proof. Let U_1 and U_2 be open sets in \mathbb{C} which are compactly contained in Ω_1 and Ω_2 respectively such that $U_1 \cap U_2 \neq \emptyset$. Consider the auxiliary metric

$$\lambda(z) |dz| := \frac{\lambda_{U_1}(z) \cdot \lambda_{U_2}(z)}{\lambda_{U_1 \cup U_2}(z)} |dz|$$

on $U_1 \cap U_2$. The curvature

$$\kappa_{\lambda}(z) = -\frac{\Delta \log \lambda(z)}{\lambda(z)^2}$$

of this metric is

$$\kappa_{\lambda}(z) = -\left(\frac{\lambda_{U_1 \cup U_2}(z)}{\lambda_{U_1}(z)}\right)^2 - \left(\frac{\lambda_{U_1 \cup U_2}(z)}{\lambda_{U_2}(z)}\right)^2 + \left(\frac{\lambda_{U_1 \cup U_2}(z)^2}{\lambda_{U_1}(z) \cdot \lambda_{U_2}(z)}\right)^2.$$

Since

$$\lambda_{U_1 \cup U_2}(z) \leq \lambda_{U_j}(z), \quad j = 1, 2, \quad (2.6)$$

by Lemma 2.2, we deduce $\kappa_{\lambda}(z) \geq -2$ for all $z \in U_1 \cap U_2$. Using again (2.6) and Lemma 2.2, we also see that

$$\lim_{z \rightarrow \xi} \lambda(z) = +\infty \quad \text{for every } \xi \in \partial(U_1 \cap U_2).$$

Since $U_1 \cap U_2$ is compactly contained in $\Omega_1 \cap \Omega_2$, the continuous function $\lambda_{\Omega_1 \cap \Omega_2} : \Omega_1 \cap \Omega_2 \rightarrow \mathbb{R}$ is bounded on $U_1 \cap U_2$, so the function

$$u(z) := \log \left(\frac{\lambda_{\Omega_1 \cap \Omega_2}(z)}{\sqrt{2} \cdot \lambda(z)} \right), \quad z \in U_1 \cap U_2,$$

has a continuous extension to $\overline{U_1 \cap U_2}$ which vanishes on $\partial(U_1 \cap U_2)$. Now we assume that $u(z) > 0$ for some $z \in U_1 \cap U_2$. Then u attains its maximal value at some point $z_0 \in U_1 \cap U_2$. This implies

$$0 \geq \Delta u(z_0) = \Delta \log \lambda_{\Omega_1 \cap \Omega_2}(z_0) - \Delta \log \lambda(z_0) \geq \lambda_{\Omega_1 \cap \Omega_2}(z_0)^2 - 2\lambda(z_0)^2,$$

so $\lambda_{\Omega_1 \cap \Omega_2}(z_0)/(\sqrt{2} \cdot \lambda(z_0)) \leq 1$, that is, $u(z_0) \leq 0$, a contradiction. It follows that $u(z) \leq 0$ for all $z \in U_1 \cap U_2$, i.e.,

$$\lambda_{U_1}(z) \cdot \lambda_{U_2}(z) \geq \frac{1}{\sqrt{2}} \cdot \lambda_{U_1 \cup U_2}(z) \cdot \lambda_{\Omega_1 \cap \Omega_2}(z) \quad \text{for all } z \in U_1 \cap U_2.$$

This inequality holds for all open sets U_1 and U_2 which are compactly contained in Ω_1 resp. Ω_2 . An application of Lemma 2.5 and Lemma 2.6 therefore completes the proof of Lemma 2.7. \square

We are now in a position to prove the inequality of Theorem 2.1 for the case that Ω_1 and Ω_2 are bounded by finitely many analytic arcs.

Lemma 2.8

Let Ω_1 and Ω_2 be open sets in \mathbb{C} such that $\Omega_1 \cap \Omega_2 \neq \emptyset$ and $\Omega_1 \cup \Omega_2$ is hyperbolic. Suppose that $\partial\Omega_1$ and $\partial\Omega_2$ consists of finitely many analytic Jordan curves. Then

$$\lambda_{\Omega_1}(z) \cdot \lambda_{\Omega_2}(z) \geq \lambda_{\Omega_1 \cup \Omega_2}(z) \cdot \lambda_{\Omega_1 \cap \Omega_2}(z) \quad \text{for all } z \in \Omega_1 \cap \Omega_2.$$

Proof. We consider the auxiliary function

$$u(z) := \log^+ \left(\frac{\lambda_{\Omega_1 \cap \Omega_2}(z) \cdot \lambda_{\Omega_1 \cup \Omega_2}(z)}{\lambda_{\Omega_1}(z) \cdot \lambda_{\Omega_2}(z)} \right), \quad z \in \Omega_1 \cap \Omega_2.$$

Here, $\log^+ x := \max\{0, \log x\}$ for every positive real number x , so by definition, u is non-negative. We need to show $u(z) \equiv 0$.

(i) We first prove that u is subharmonic on $\Omega_1 \cap \Omega_2$. For this purpose we note that the monotonicity principle of the hyperbolic metric (Lemma 2.2) shows that

$$\lambda_{\Omega_1}(z) \geq \lambda_{\Omega_1 \cup \Omega_2}(z), \quad \lambda_{\Omega_2}(z) \geq \lambda_{\Omega_1 \cup \Omega_2}(z) \quad \text{for all } z \in \Omega_1 \cap \Omega_2. \quad (2.7)$$

Now fix a point $z_0 \in \Omega_1 \cap \Omega_2$ such that $u(z_0) > 0$. Then we can compute $\Delta u(z_0)$ using the curvature equation (1.1). This gives us

$$\begin{aligned} \Delta u(z_0) &= \lambda_{\Omega_1 \cap \Omega_2}(z_0)^2 + \lambda_{\Omega_1 \cup \Omega_2}(z_0)^2 - (\lambda_{\Omega_1}(z_0)^2 + \lambda_{\Omega_2}(z_0)^2) \\ &= \left(\lambda_{\Omega_1 \cap \Omega_2}(z_0) - \lambda_{\Omega_1 \cup \Omega_2}(z_0) \right)^2 - \left(\lambda_{\Omega_1}(z_0) - \lambda_{\Omega_2}(z_0) \right)^2 \\ &\quad - 2 \left(\lambda_{\Omega_1}(z_0) \lambda_{\Omega_2}(z_0) - \lambda_{\Omega_1 \cap \Omega_2}(z_0) \lambda_{\Omega_1 \cup \Omega_2}(z_0) \right). \end{aligned} \quad (2.8)$$

Now observe that $u(z_0) > 0$ is the same as $\lambda_{\Omega_1}(z_0)\lambda_{\Omega_2}(z_0) - \lambda_{\Omega_1 \cap \Omega_2}(z_0)\lambda_{\Omega_1 \cup \Omega_2}(z_0) < 0$, so

$$\Delta u(z_0) > \left(\lambda_{\Omega_1 \cap \Omega_2}(z_0) - \lambda_{\Omega_1 \cup \Omega_2}(z_0)\right)^2 - \left(\lambda_{\Omega_1}(z_0) - \lambda_{\Omega_2}(z_0)\right)^2.$$

We claim that $\Delta u(z_0) \geq 0$. In fact, if $\Delta u(z_0) < 0$, then

$$\left(\lambda_{\Omega_1}(z_0) - \lambda_{\Omega_2}(z_0)\right)^2 > \left(\lambda_{\Omega_1 \cap \Omega_2}(z_0) - \lambda_{\Omega_1 \cup \Omega_2}(z_0)\right)^2.$$

Since $\lambda_{\Omega_1 \cap \Omega_2}(z_0) \geq \lambda_{\Omega_1 \cup \Omega_2}(z_0)$ by Lemma 2.2 and since we may assume $\lambda_{\Omega_1}(z_0) \geq \lambda_{\Omega_2}(z_0)$, we get

$$\lambda_{\Omega_1}(z_0) - \lambda_{\Omega_2}(z_0) > \lambda_{\Omega_1 \cap \Omega_2}(z_0) - \lambda_{\Omega_1 \cup \Omega_2}(z_0).$$

This however contradicts the monotonicity property of the hyperbolic metric, which in particular says that $\lambda_{\Omega_1}(z_0) \leq \lambda_{\Omega_1 \cap \Omega_2}(z_0)$ and $\lambda_{\Omega_1 \cup \Omega_2}(z_0) \leq \lambda_{\Omega_2}(z_0)$. We have therefore shown that $\Delta u(z_0) \geq 0$ for every $z_0 \in \Omega_1 \cap \Omega_2$ such that $u(z_0) > 0$. But since u is non-negative by definition, this easily implies that u is subharmonic on $\Omega_1 \cap \Omega_2$.

(ii) We now examine the boundary behaviour of the auxiliary function u and fix a point $\xi \in \partial(\Omega_1 \cap \Omega_2)$. Then $\xi \in \partial\Omega_1 \cup \partial\Omega_2$. Let us consider the case $\xi \in \partial\Omega_1 \setminus \partial\Omega_2$. Then we can apply Lemma 2.4, and this gives

$$\lim_{z \rightarrow \xi} \frac{\lambda_{\Omega_1}(z)}{\lambda_{\Omega_1 \cap \Omega_2}(z)} = 1. \quad (2.9)$$

Hence, using the second inequality of (2.7), we get that

$$\liminf_{z \rightarrow \xi} \frac{\lambda_{\Omega_1}(z) \cdot \lambda_{\Omega_2}(z)}{\lambda_{\Omega_1 \cap \Omega_2}(z) \cdot \lambda_{\Omega_1 \cup \Omega_2}(z)} \geq 1.$$

We have proved this inequality if $\xi \in \partial(\Omega_1 \cap \Omega_2)$ belongs to $\partial\Omega_1 \setminus \partial\Omega_2$. Switching the roles of Ω_1 and Ω_2 , we see that this estimate also holds if $\xi \in \partial\Omega_2 \setminus \partial\Omega_1$, i.e. it holds for every $\xi \in \partial(\Omega_1 \cap \Omega_2)$ such that $\xi \notin \partial\Omega_1 \cap \partial\Omega_2$. This means that the auxiliary function u has the property that

$$\limsup_{z \rightarrow \xi} u(z) = 0 \text{ for every } \xi \in \partial(\Omega_1 \cap \Omega_2) \setminus (\partial\Omega_1 \cap \partial\Omega_2).$$

Since analytic Jordan curves can only intersect at finitely many points, we deduce that

$$\limsup_{z \rightarrow \xi} u(z) = 0 \quad \text{for all but finitely many } \xi \in \partial(\Omega_1 \cap \Omega_2).$$

(iii) Finally we note that the auxiliary function u is bounded above by $\log \sqrt{2}$ (Lemma 2.7). Therefore we are in a position to apply the extended maximum principle for subharmonic functions (see [19, Theorem 3.6.9]), and this implies that $u(z) \leq 0$ for all $z \in \Omega_1 \cap \Omega_2$. \square

Proof of Theorem 2.1. The estimate (2.1) of Theorem 2.1 follows immediately from Lemma 2.8, Lemma 2.5 and Lemma 2.6, so it remains to deal with the case of equality. We consider the function

$$u(z) := \log \left(\frac{\lambda_{\Omega_1 \cap \Omega_2}(z) \cdot \lambda_{\Omega_1 \cup \Omega_2}(z)}{\lambda_{\Omega_1}(z) \cdot \lambda_{\Omega_2}(z)} \right), \quad z \in \Omega_1 \cap \Omega_2.$$

Note that we have already proven that $u(z) \leq 0$ in $\Omega_1 \cap \Omega_2$. As in the proof of Lemma 2.8, we have

$$\begin{aligned} \Delta u(z) &= \left(\lambda_{\Omega_1 \cap \Omega_2}(z) - \lambda_{\Omega_1 \cup \Omega_2}(z) \right)^2 - \left(\lambda_{\Omega_1}(z) - \lambda_{\Omega_2}(z) \right)^2 \\ &\quad + 2 \left(\lambda_{\Omega_1 \cap \Omega_2}(z) \lambda_{\Omega_1 \cup \Omega_2}(z) - \lambda_{\Omega_1}(z) \lambda_{\Omega_2}(z) \right). \end{aligned} \quad (2.10)$$

Now, Lemma 2.2 implies $\lambda_{\Omega_1 \cap \Omega_2}(z) \geq \lambda_{\Omega_j}(z) \geq \lambda_{\Omega_1 \cup \Omega_2}(z)$ for $j = 1, 2$, so

$$\lambda_{\Omega_1 \cap \Omega_2}(z) - \lambda_{\Omega_1 \cup \Omega_2}(z) \geq |\lambda_{\Omega_1}(z) - \lambda_{\Omega_2}(z)|.$$

Inserting this into (2.10), we get

$$\Delta u(z) \geq 2 \left(\lambda_{\Omega_1 \cap \Omega_2}(z) \lambda_{\Omega_1 \cup \Omega_2}(z) - \lambda_{\Omega_1}(z) \lambda_{\Omega_2}(z) \right). \quad (2.11)$$

Applying the elementary inequality

$$a \log \frac{b}{a} \leq b - a \text{ for all } a, b \in \mathbb{R}, a \geq b > 0$$

for $a = \lambda_{\Omega_1}(z) \lambda_{\Omega_2}(z)$ and $b = \lambda_{\Omega_1 \cap \Omega_2}(z) \cdot \lambda_{\Omega_1 \cup \Omega_2}(z)$, we deduce from (2.11) that

$$\Delta u(z) \geq 2 \lambda_{\Omega_1}(z) \lambda_{\Omega_2}(z) \cdot u(z), \quad z \in \Omega_1 \cap \Omega_2.$$

Hence, the strong maximum principle of Hopf (see [12, Thm 2.1.2]) implies that in each connected component of $\Omega_1 \cap \Omega_2$ either $u = 0$ or $u < 0$. Therefore, if $z_0 \in \Omega_1 \cap \Omega_2$ is a point such that equality holds in (2.1), then equality holds in (2.1) for all z in $(\Omega_1 \cap \Omega_2)(z_0)$, the component of $\Omega_1 \cap \Omega_2$ which contains z_0 . In view of (2.10) this implies

$$\left(\lambda_{\Omega_1 \cap \Omega_2}(z) - \lambda_{\Omega_1 \cup \Omega_2}(z) \right)^2 = \left(\lambda_{\Omega_1}(z) - \lambda_{\Omega_2}(z) \right)^2, \quad z \in (\Omega_1 \cap \Omega_2)(z_0). \quad (2.12)$$

We need to show that $\Omega_1(z_0) \subseteq \Omega_2(z_0)$ or $\Omega_2(z_0) \subseteq \Omega_1(z_0)$. This is obviously true if $(\Omega_1 \cap \Omega_2)(z_0) = (\Omega_1 \cup \Omega_2)(z_0)$, so we need to analyze the case $(\Omega_1 \cap \Omega_2)(z_0) \subsetneq (\Omega_1 \cup \Omega_2)(z_0)$. Since the hyperbolic metric is *strictly* decreasing with expanding domains (see [8, p. 683]), we have $\lambda_{\Omega_1 \cap \Omega_2} > \lambda_{\Omega_1 \cup \Omega_2}$ in $(\Omega_1 \cap \Omega_2)(z_0)$ and therefore the identity (2.12) and the continuity of $\lambda_{\Omega_1} - \lambda_{\Omega_2}$ imply that there either $\lambda_{\Omega_1} > \lambda_{\Omega_2}$ or $\lambda_{\Omega_1} < \lambda_{\Omega_2}$. In the first case, we can deduce from (2.12) that

$$\lambda_{\Omega_1 \cap \Omega_2}(z) - \lambda_{\Omega_1 \cup \Omega_2}(z) = \lambda_{\Omega_1}(z) - \lambda_{\Omega_2}(z), \quad z \in (\Omega_1 \cap \Omega_2)(z_0). \quad (2.13)$$

This shows $(\Omega_1 \cap \Omega_2)(z_0) = \Omega_1(z_0)$, because otherwise $\lambda_{\Omega_1 \cap \Omega_2}(z) > \lambda_{\Omega_1}(z)$, i.e., $\lambda_{\Omega_1 \cup \Omega_2}(z) = \lambda_{\Omega_1 \cap \Omega_2}(z) - \lambda_{\Omega_1}(z) + \lambda_{\Omega_2}(z) > \lambda_{\Omega_2}(z)$, a contradiction. Hence we get $\lambda_{\Omega_1 \cap \Omega_2}(z) = \lambda_{\Omega_1}(z)$ for every $z \in (\Omega_1 \cap \Omega_2)(z_0)$. Therefore, (2.13) shows $\lambda_{\Omega_1 \cup \Omega_2} = \lambda_{\Omega_2}$ on $(\Omega_1 \cap \Omega_2)(z_0)$, so $(\Omega_1 \cup \Omega_2)(z_0) = \Omega_2(z_0)$. Putting all this together gives $\Omega_1(z_0) = (\Omega_1 \cap \Omega_2)(z_0) \subsetneq (\Omega_1 \cup \Omega_2)(z_0) = \Omega_2(z_0)$. \square

Remark 2.9

The approximation technique by smooth open sets, which has been used to prove Theorem 2.1, cannot be avoided entirely. This follows from the fact that if e.g. Ω_1 is a non-smooth open set, then the crucial limit relation (2.9), which in turn is based on Lemma 2.4, is no longer valid. To see this, take $\Omega_1 := \mathbb{C} \setminus \{-1, 1\}$ and $\Omega_2 := \mathbb{D}$, so $\Omega_1 \cap \Omega_2 = \mathbb{D}$ and

$$\lambda_{\Omega_1 \cap \Omega_2}(z) = \lambda_{\mathbb{D}}(z) = \frac{2}{1 - |z|^2},$$

On the other hand, it is known (see [17]) that

$$\lim_{z \rightarrow 1} \lambda_{\mathbb{C} \setminus \{-1, 1\}}(z) |z - 1| \log \left(\frac{1}{|z - 1|} \right) = 1.$$

Hence,

$$\lim_{z \rightarrow 1} \frac{\lambda_{\Omega_1}(z)}{\lambda_{\Omega_1 \cap \Omega_2}(z)} = \lim_{z \rightarrow 1} \frac{\lambda_{\mathbb{C} \setminus \{-1, 1\}}(z)}{\lambda_{\mathbb{D}}(z)} = 0.$$

It would be interesting to see whether the regularity conditions imposed on the boundary set Γ in Lemma 2.4 can be considerably weakened.

3 Concluding remark

We conclude this paper by noting that Theorem 1.1 and Theorem 2.1 remain valid, for subdomains (or open subsets) of a Riemann surface since it makes sense to speak of the value of the *quotient* of two conformal metrics at a specific point on a Riemann surface, see again [17]. Hence the “Riemann surface version” of Theorem 1.1 takes the following form.

Theorem 3.1

Let R be a Riemann surface and let $\Omega_1, \Omega_2 \subseteq R$ be domains such that $\Omega_1 \cap \Omega_2 \neq \emptyset$ and $\Omega_1 \cup \Omega_2$ is hyperbolic. Then

$$\frac{\lambda_{\Omega_1}(z) \cdot \lambda_{\Omega_2}(z)}{\lambda_{\Omega_1 \cup \Omega_2}(z) \cdot \lambda_{\Omega_1 \cap \Omega_2}(z)} \geq 1 \quad \text{for all } z \in \Omega_1 \cap \Omega_2.$$

If equality holds for one point $z \in \Omega_1 \cap \Omega_2$, then $\Omega_1 \subseteq \Omega_2$ or $\Omega_2 \subseteq \Omega_1$. In this case, equality holds for all points in $\Omega_1 \cap \Omega_2$.

The proof is almost identical to the proof of Theorem 1.1 with obvious modifications.

References

- [1] S. Agard, Distortion theorems for quasiconformal mappings, *Ann. Acad. Sci. Fenn. Ser. A I* (413), 12 pp (1968).
- [2] L. Ahlfors, An extension of Schwarz's lemma, *Trans. Amer. Math. Soc.* **42** (1938), 359–364.
- [3] A. Baernstein II and A. Yu. Solynin, Monotonicity and comparison results for conformal invariants, *Rev. Mat. Iberoam.* (2013), **29** No. 1, 91–113.
- [4] A. F. Beardon, Ch. Pommerenke, The Poincaré metric of plane domains, *J. London Math. Soc.* **18** (2), 475–483 (1978).
- [5] D. Betsakos, Estimation of the hyperbolic metric by using the punctured plane, *Math. Z.* **259** (2008), no. 1, 187–196.
- [6] G. Choquet, Theory of capacities, *Ann. Inst. Fourier Grenoble* **5**, 131–295 (1953–1954).
- [7] F. P. Gardiner, N. Lakic, Comparing Poincaré densities, *Ann. Math.* **154** (2), 245–267 (2001).
- [8] W. K. Hayman, *Subharmonic functions*, Vol. 2, London Math. Soc., 1989.
- [9] M. Heins, On a class of conformal metrics, *Nagoya Math. J.* (1962), **21**, 1–60.
- [10] D. A. Hejhal, Universal covering maps for variable regions, *Math. Z.* **137** (1974), 7–20.
- [11] J. A. Hempel, The Poincaré metric on the twice punctured plane and the theorems of Landau and Schottky, *J. London Math. Soc.* **20** (2), 435–445 (1979).
- [12] J. Jost, *Partial Differential Equations*, Springer, 2012.
- [13] D. Kraus, O. Roth and St. Ruscheweyh, A boundary version of Ahlfors' Lemma, locally complete conformal metrics and conformally invariant reflection principles for analytic maps, *J. Anal. Math.* **101** (2007), 219–256.
- [14] D. Kraus, O. Roth and T. Sugawa, Metrics with conical singularities on the sphere and sharp extensions of the theorems of Landau and Schottky, *Math. Z.* **267** (2011), no. 3–4, 851–868.
- [15] D. Minda, Bloch constants, *J. Anal. Math.* **41** (1982), 54–84.
- [16] D. Minda, The strong form of Ahlfors' lemma, *Rocky Mountain J. Math.* **17** no. 3 (1987), 457–461.

- [17] D. Minda, The density of the hyperbolic metric near an isolated boundary point, *Compl. Var.* **32** (1997), 331–340.
- [18] Ch. Pommerenke, *Boundary behaviour of conformal maps*, Springer, 1992.
- [19] T. J. Ransford, *Potential theory in the complex plane*, Cambridge Univ. Press, 1995.
- [20] Renggli, An inequality for logarithmic capacities, *Pacific J. Math.* **11** No. 1 (1961), 313–314.
- [21] A. Yu. Solynin, Ordering of sets, hyperbolic metrics, and harmonic measures, *J. Math. Sci.* **95** no. 3, 2256–2266 (1999).
- [22] A. Yu. Solynin, Elliptic operators and Choquet capacities, *J. Math. Sci.* **166** no. 2, 210–213 (2010).
- [23] A. Yu. Solynin, M. Vuorinen, Estimates for the hyperbolic metric of the punctured plane and applications, *Israel J. Math.* **124**, 29–60 (2001).
- [24] T. Sugawa, M. Vuorinen, Some inequalities for the Poincaré metric of plane domains, *Math. Z.* **250**, 885–906 (2005).
- [25] K. Strebel, *Vorlesungen über Riemannsche Flächen*, Vandenhoeck & Ruprecht, 1980.
- [26] A. Weitsman, A symmetry property of the Poincaré metric, *Bull. London Math. Soc.* **11**, 295–299 (1979).

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